

Last time: Sets and Set Operations

Venn Diagrams $\leftarrow$ (Just for Intuition...
NOT PROOFS)

Recall: For set S, T we defined:

$$S \cap T = \{x : x \in S \text{ and } x \in T\}$$

$$S \cup T = \{x : x \in S \text{ or } x \in T\}$$

$$S \setminus T = \{x : x \in S \text{ but } x \notin T\}$$

$$S \times T = \{(x, y) : x \in S \text{ and } y \in T\}$$

$$S \subseteq T \text{ when } \forall x \in S [x \in T] \equiv \boxed{\forall x [x \in S \rightarrow x \in T]}$$

$$S = T \text{ when } \forall x [x \in S \leftrightarrow x \in T] \leftarrow$$

Prop: Let S and T be sets. Then $S \cup T = T \cup S$.

pf: Let S and T be sets. $\leftarrow$ state assumptions.

(Remark: we will show $S \cup T \subseteq T \cup S$
and $T \cup S \subseteq S \cup T$)

Remarks = red.

$\leftarrow$ signals first step
To see $S \cup T \subseteq T \cup S$, we let $[x \in S \cup T \text{ be arbitrary}]$.
 $\leftarrow$ assumption for conditional part

Thus $x \in S$ or $x \in T$ by the definition of set union.
 $\leftarrow$ justification.

As a result, $x \in T$ or $x \in S$ by commutativity of "or".

Hence $x \in T \cup S$ by definition of set union. [We

now see $x \in S \cup T$ implies $x \in T \cup S$, so $S \cup T \subseteq T \cup S$
by definition of the subset relation.]

To see $T \cup S \subseteq S \cup T$, we let $x \in T \cup S$ be arbitrary.

Thus $x \in T$ or $x \in S$ by definition of set union.

We thus obtain $x \in S$ or $x \in T$ by commutativity of

"or". Hence $x \in S \cup T$ by definition of set union.
 We now see $x \in T \cup S$ implies $x \in S \cup T$, so $T \cup S \subseteq S \cup T$.
 Now we have $S \cup T \subseteq T \cup S$ and $T \cup S \subseteq S \cup T$, so
 $S \cup T = T \cup S$ as originally claimed. $\square$
 → Same pattern. Last line: Confirms (to the reader) we think we're done proving!

Point: Every good proof follows these rules:

- ① Clearly written (mathematical) English.
- ② Every claim we make comes with a justification via "previous results," definitions, and basic logic.

Rule of Thumb: Treat your reader like a compiler, and the language you use like a programming language. Must follow the established syntax and semantics!

NB: To prove two sets are equal, we prove they are mutually subsets of one another...

I.e. to prove $A = B$, we show $A \subseteq B$ and $B \subseteq A$.

Prop: Let A, B , and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

pf: Let A, B , and C be sets. Suppose $A \subseteq B$ and $B \subseteq C$.
 (WTS: $A \subseteq C$ i.e. $\forall x [x \in A \rightarrow x \in C]$)
 ↑ "want to show"

Let $x \in A$ be arbitrary. (WTS: $x \in C$).

As $A \subseteq B$ and $x \in A$, we have $x \in B$ by definition of the subset relation.

As $B \subseteq C$ and $x \in B$, we have $x \in C$ by definition of the subset relation.

$\left(\begin{array}{l} \forall y [y \in A \rightarrow y \in B] \\ x \in A \quad \therefore x \in B \\ \text{"Quantified Modus ponens"} \end{array} \right)$

As $x \in A$ was arbitrary, we have $A \subseteq C$ by definition of the subset relation. $\square$

Prop: For all sets A, B , and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

pf: Let A, B , and C be sets.

($\subseteq$): Let $x \in A \cap (B \cup C)$ be arbitrary. By definition of intersection we have $x \in A$ and $x \in B \cup C$. By definition of set union we have both $x \in A$ and either $x \in B$ or $x \in C$.

Thus, by distribution of "and" over "or", we have either both $x \in A$ and $x \in B$ or both $x \in A$ and $x \in C$. Thus

(translates: $[x \in A] \wedge ([x \in B] \vee [x \in C])$)

we have either $x \in A \cap B$ or $x \in A \cap C$ by definition of set intersection. Hence $x \in (A \cap B) \cup (A \cap C)$ by definition of set union. As x was arbitrary, we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

($\supseteq$): Let $x \in (A \cap B) \cup (A \cap C)$ be arbitrary. Thus either $x \in A \cap B$ or $x \in A \cap C$ by definition of set union. By definition of set intersection, either

both $x \in A$ and $x \in B$ or both $x \in A$ and $x \in C$.

Thus both $x \in A$ and either $x \in B$ or $x \in C$ by distribution of "and" over "or". Thus both $x \in A$ and $x \in B \cup C$ by definition of set union.

Finally, $x \in A \cap (B \cup C)$ by definition of set intersection.

As x was arbitrary, we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence as both $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, we have shown

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ as desired. $\square$

$$\forall x \in A [x \in B] \equiv \forall x [(x \in A) \rightarrow (x \in B)]$$

Recall: $\mathcal{P}(S) = \{T : T \subseteq S\}$ $\leftarrow$

Prop: Let A and B be sets.

If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

pf: Let A and B be sets and suppose $A \subseteq B$.

Let $X \in \mathcal{P}(A)$ be arbitrary. By definition of the powerset, we have $X \subseteq A$; on the other hand, $A \subseteq B$, so $X \subseteq B$ by a *previous proposition.

Hence $X \in \mathcal{P}(B)$ by definition of powerset. As $X \in \mathcal{P}(A)$ was arbitrary, we have $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ as claimed. $\square$

Propositions to try proving for homework:

Let A, B, C, D be sets.

① $A \cap B = B \cap A$.

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

③ $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

↳ Q: Does $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$?

④ Is $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$?

Is $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$?